Announcements

1) HW #4 due Thursday

2) Quiz Tuesday next week

Lagrange Multiplier Example

Arithmetic-Geometric Mean Inequality:

If $x_1, x_2, \cdots, x_n$ are positive real numbers,

$$\boxed{(x_1 x_2 x_3 \cdots x_n)^{1/n} \leq \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}}$$

For $n=3$

$$(xyz)^{1/3} \le \frac{x+y+z}{3}$$

Why is this true?

Maximize $(xyz)^{1/3}$ – same as maximizing xyz.

Want to maximize subject to the constraint

$$\frac{x+y+z}{3} = k$$, same

as $x+y+z = 3k$.

Using calculus

Maximize $f(x,y,z) = xyz$

Subject to $3k = x+y+z = g(x,y,z)$

Lagrange Multipliers

$$\nabla f = \langle yz, xz, xy \rangle$$

$$\nabla g = \langle 1, 1, 1 \rangle$$

Set

$$\nabla f = \lambda \nabla g$$

$$\langle yz, xz, xy \rangle = \lambda \langle 1, 1, 1 \rangle$$

$$= \langle \lambda, \lambda, \lambda \rangle$$

We get

$$\lambda = xy = yz = xz$$

$x = z$ if $y \neq 0$. but $y > 0$!

$x = y$ since $z > 0$.

Then $x = y = z$.

Use constraint:

$$x + y + z = 3k$$

$$3x = 3k,$$

$$x = k, \quad y = k, \quad z = k$$

We found that

$f(x,y,z) = xyz$, subject

to $\frac{x+y+z}{3} = k$, has

a maximum when

$x = y = z = k$. In this

case $f(x,y,z) = k^3$

$$(xyz)^{1/3} = (k^3)^{1/3} = k$$

$$= \frac{x+y+z}{3}$$

This says:

$$(xyz)^{1/3} = \frac{x+y+z}{3}$$

when $x=y=z$;

otherwise, we have strict inequality!

Integration

(Chapter 15)

We've seen two heads of the Calculus Cerberus in two or three variables – limits and differentiation.

In One Dimension

Two types of regions:

intervals ($\int_a^b f(x)\,dx$ when $a < b$ - the interval is $[a,b]$) or points

($\int_a^a f(x)\,dx = 0$)

Q: What geometric concept does the definite integral capture?

A: AREA (under a curve with non-negative y-values)

Definition: (one variable integral)

$$\int_a^b f(x)\,dx$$

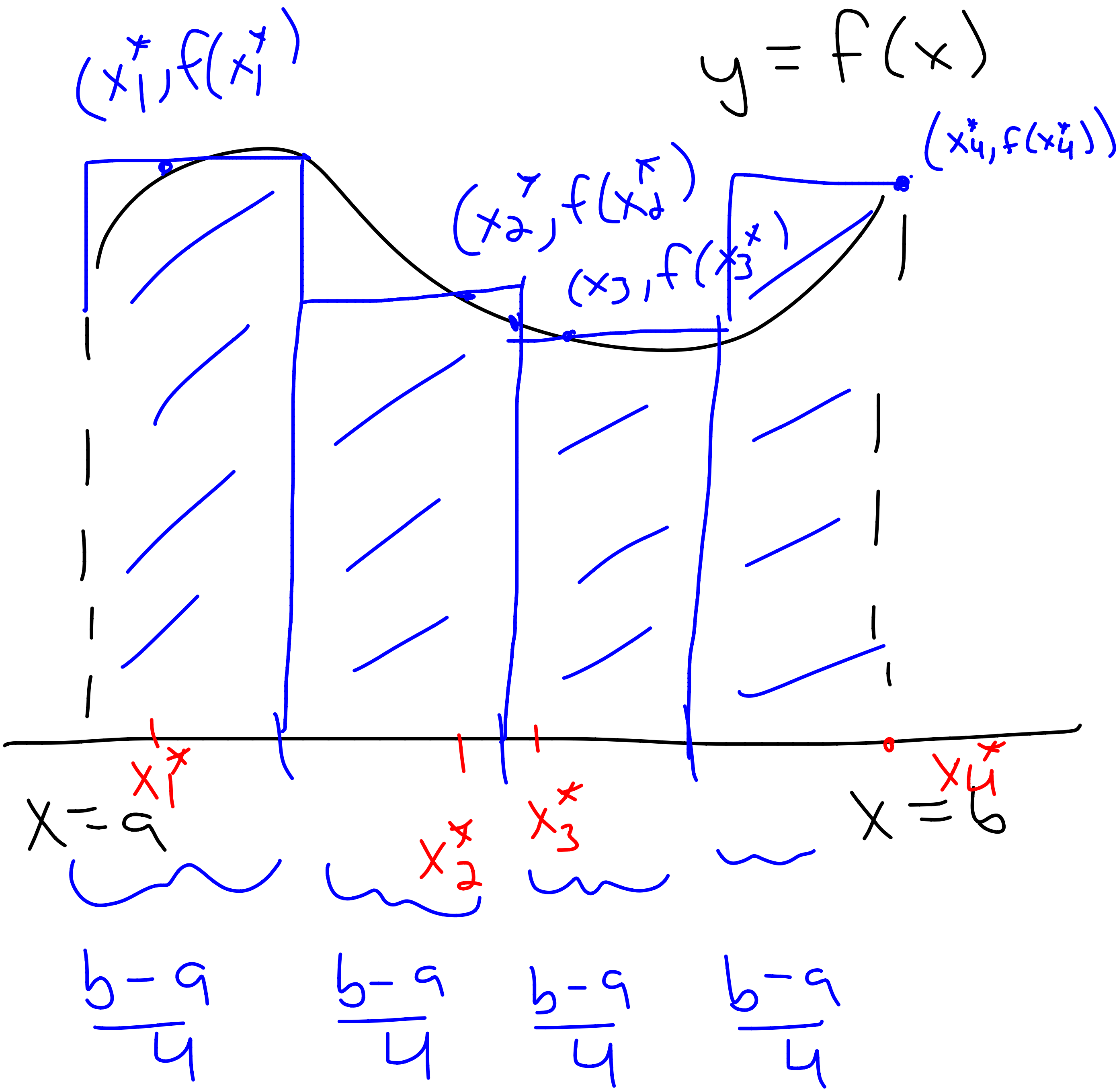
$$= \lim_{n\to\infty}\left(\frac{b-a}{n}\sum_{i=1}^{n} f(x_i^*)\right)$$

where x_i^* is a point in the interval $\left[a+\frac{(i-1)(b-a)}{n},\ a+\frac{i(b-a)}{n}\right]$

provided the limit exists!

The definite integral exists for all continuous functions f.

Picture



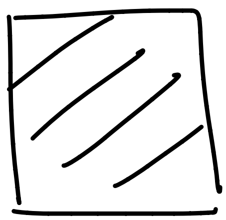
Approximate using rectangles, make the width go to zero.

So in three dimensions

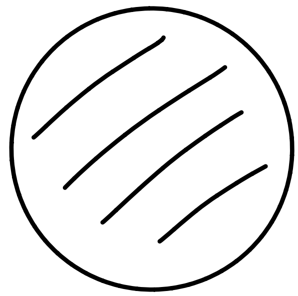
$z = f(x,y)$, the graph is in three dimensions.

If $f(x,y) > 0$, the integral over a region should give volume — provided the region is non-degenerate! So no points or one-dimensional regions (lines, curves, etc.)

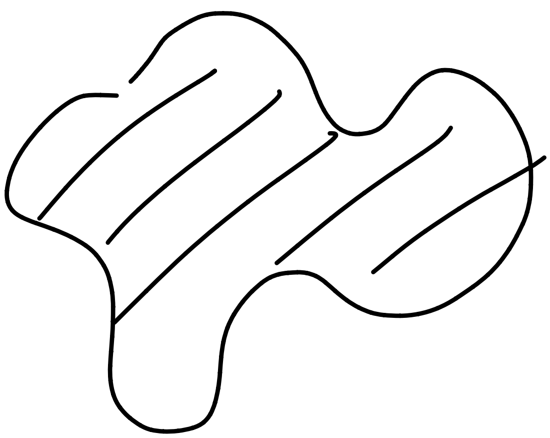
Good regions of integration



rectangles



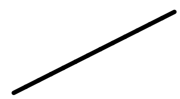
circles



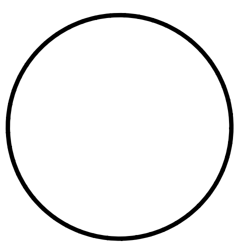
weird 2-D region

Bad Regions of Integration

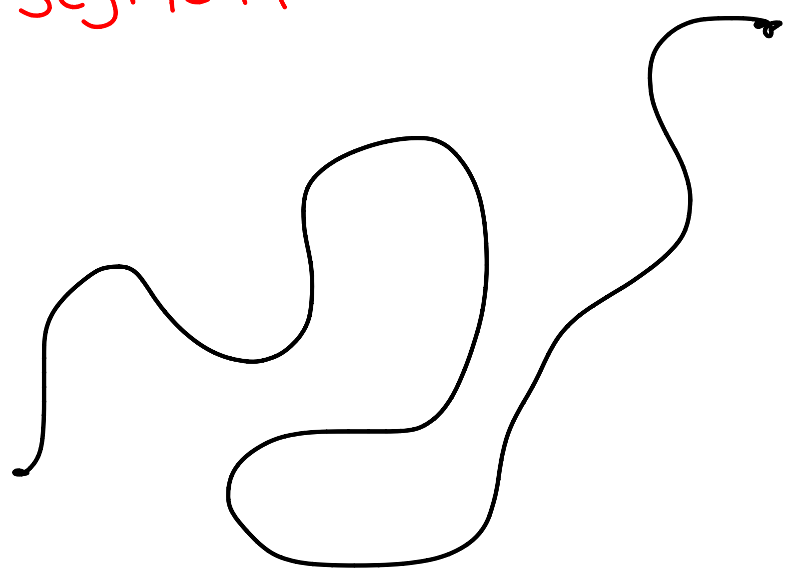
point



line segment



circle with no interior



curve

The Definite Integral

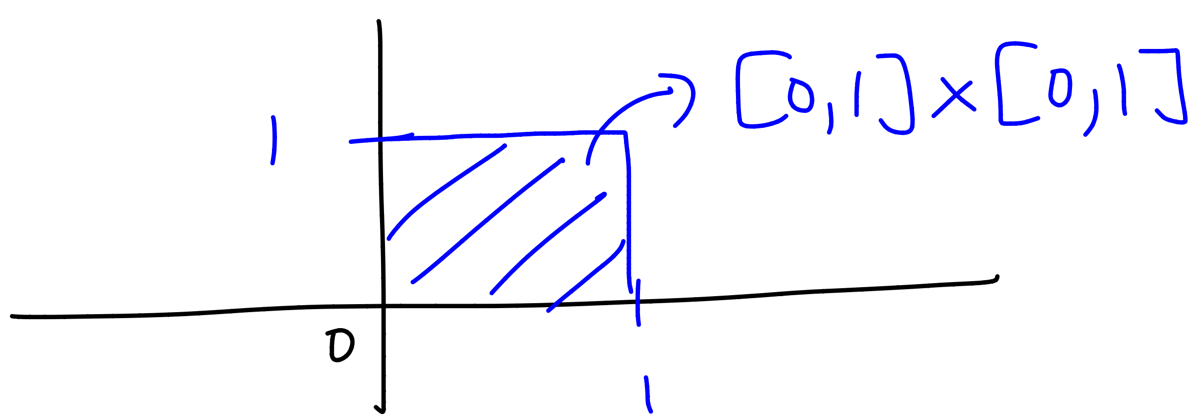
($\mathbb{R}^2$, rectangle)

A solid rectangle in $\mathbb{R}^2$ is denoted by

$$[a,b] \times [c,d]$$

$$= \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$$

Example $[0,1] \times [0,1]$ is



Subdivide $[a,b]$ into

n intervals of size $\frac{b-a}{n}$.

Subdivide $[c,d]$ into

m intervals of size $\frac{d-c}{m}$

Pick a point $(x_{i,j}^*, y_{i,j}^*)$

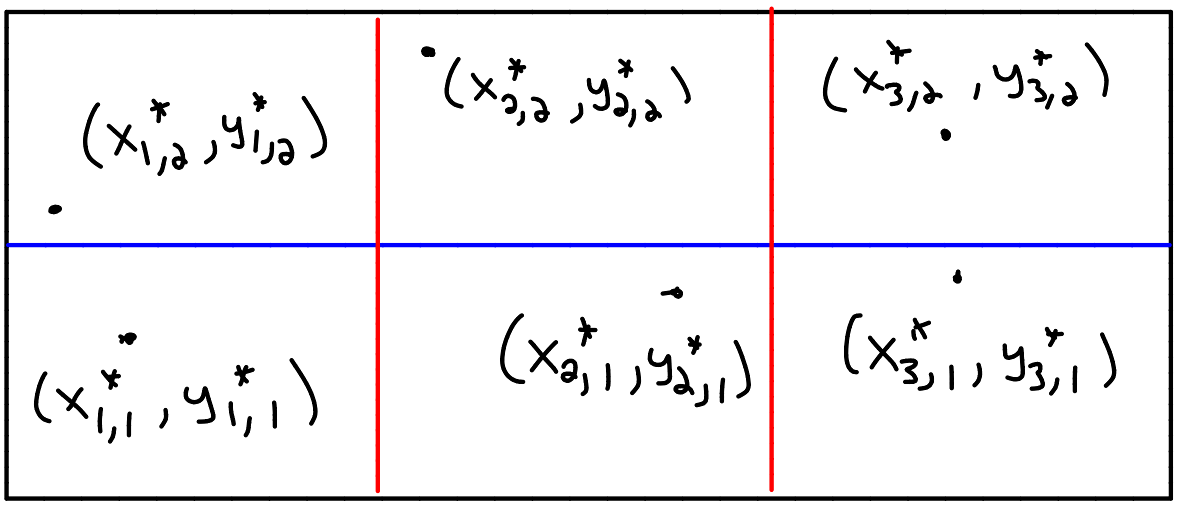
in the rectangle

$$\left[a+\frac{(i-1)(b-a)}{n}, a+\frac{i(b-a)}{n}\right] \times \left[c+\frac{(j-1)(d-c)}{m}, c+\frac{j(d-c)}{m}\right]$$

for each pair (i,j) with

$1 \leq j \leq m, \quad 1 \leq i \leq n$.

Picture



$$[0,3] \times [1,2]$$

$$m = 2, \quad n = 3$$

Horrid Definition For

$z = f(x,y)$ defined on a rectangle $[a,b] \times [c,d]$ in $\mathbb{R}^2$, the definite integral of f over $[a,b] \times [c,d]$ is denoted by

$$\int_{[a,b]\times[c,d]} f(x,y)\, dA$$

A for "area"

and defined as

$$\boxed{\begin{aligned}&\iint_{[a,b]\times[c,d]} f(x,y)\,dA \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{b-a}{n}\frac{d-c}{m}\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{i,j}^{*}, y_{i,j}^{*})\right)\end{aligned}}$$

provided the limit exists!

Again, if f is continuous, the limit will exist. How do you find it?

Fubini's Theorem

If $z = f(x,y)$ is continuous on $[a,b] \times [c,d]$, then

$$\boxed{\begin{aligned} &\iint\limits_{[a,b]\times[c,d]} f(x,y)\,dA \\ &= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx \\ &= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy \end{aligned}}$$

where

$$\int_a^b f(x,y)\,dx$$ means treat

$f(x,y)$ as a function of x with y constant, integrate with respect to x. For

$$\int_c^d f(x,y)\,dy,$$ interchange the roles of x and y.

Example 1:

$$\iint_{[0,\ln(2)]\times[1,4]} y e^{xy} \, dA$$

Use Fubini's Theorem:

$$\iint_{[0,\ln(2)]\times[1,4]} y e^{xy} \, dA$$

$$= \int_0^{\ln(2)} \Big(\underbrace{\int_1^4 y e^{xy} \, dy}_{\text{integration by parts!}} \Big) dx$$

The integral is also equal to

$$\int_1^4 \left(\int_0^{\ln(2)} y e^{xy} \, dx \right) dy$$

$$= \int_1^4 \left(y \int_0^{\ln(2)} e^{xy} \, dx \right) dy$$

(since y is a constant w.r.t. x)

$$= \int_1^4 \left(y \left(\frac{e^{xy}}{y} \Big|_0^{\ln(2)} \right) \right) dy$$

$$= \int_1^4 \cancel{y/y} \left(e^{\ln(2) y} - 1 \right) dy$$

$$= \int_1^4 \left(e^{\ln(2) y} - 1 \right) dy$$

$$= \left(\frac{e^{\ln(2) y}}{\ln(2)} - y \right) \Bigg|_1^4$$

$$= \left(\frac{e^{4\ln(2)}}{\ln(2)} - 4 \right) - \left(\frac{e^{\ln(2)}}{\ln(2)} - 1 \right)$$

$$= \boxed{\frac{14}{\ln(2)} - 3}$$

Easy Scenario

If $z = f(x,y) = g(x)h(y)$ on $[a,b] \times [c,d]$ and f is continuous,

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dA$$

$$= \boxed{\int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy}$$

Example 2 $\displaystyle\iint_{[-3,3]\times[0,\pi/3]} x^3 \arctan(\ln(\cos(y)))\,dA$

By Fubini's theorem, this integral is equal to

$$\underbrace{\int_{-3}^{3} x^3\,dx}_{} \cdot \int_{0}^{\pi/3} \arctan(\ln(\cos(y)))\,dy$$

$= 0$ since x^3 is odd and the domain is symmetric.

So the value of the integral is zero!

Note: I have no idea how to integrate $\arctan(\ln(\cos(y)))$.

Boring Properties of the Integral

Let R be the region $[a,b] \times [c,d]$ in $\mathbb{R}^2$, let f, g be continuous real-valued functions on R. Then

1) $$\iint_R (f(x,y) + g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

(integrals distribute over addition)

2) If c is any constant,

$$\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$$

(can pull constants out of integrals)

3) If D is a one or zero dimensional region in $\mathbb{R}^2$, then

$$\iint_D f(x,y)\, dA = 0$$

Fake Fubini Justification

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_a^b \lim_{m\to\infty} \left(\frac{d-c}{m} \sum_{j=1}^{m} f(x, y_j^*) \right) dx$$

$$= \lim_{m\to\infty} \frac{d-c}{m} \int_a^b \sum_{j=1}^{m} f(x, y_j^*)\,dx$$

$$= \lim_{m\to\infty} \frac{d-c}{m} \sum_{j=1}^{m} \int_a^b f(x, y_j^*)\,dx$$

$$= \lim_{m\to\infty} \frac{d-c}{m} \sum_{j=1}^{m} \lim_{n\to\infty}\left(\frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*, y_j^*)\right)$$

$$= \lim_{m\to\infty} \lim_{n\to\infty} \frac{d-c}{m} \frac{b-a}{n} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)$$

$$= \iint_{[a,b]\times[c,d]} f(x,y)\, dA$$

Where did I lie to you?